$3+5=8$	Yesterday it was +3. Today the temperature is 5 degrees warmer and is 8 degrees.
$(-3)+5=2$	Yesterday it was −3 degrees. Today it is 5 degrees warmer, that is, +2.
$3+(-5)=-2$	Yesterday was +3, today it is 5 degrees colder, that is, −2.
$(-3)+(-5)=(-8)$	Yesterday was −3, today it is 5 degrees colder, that is, −8.

(Here all temperatures are measured in Celsius degrees.)

Here is another example:

$3+5=8$	Three protons + five protons = = eight protons.
$(-3)+5=2$	Three antiprotons + five protons = = two protons (ignoring γ-radiation).
$3+(-5)=-2$	Three protons + five antiprotons = = two antiprotons (ignoring γ-radiation).
$(-3)+(-5)=(-8)$	Three antiprotons + five antiprotons = = eight antiprotons.

(Protons and antiprotons are elementary particles. When a proton meets an antiproton they annihilate one another, producing gamma radiation.)

14 The multiplication of negative numbers

To find how much three times five is, you add three numbers equal to five:

$$5+5+5=15\,.$$

The same explanation may be used for the product $1\cdot 5$ if we agree that a sum having only one term is equal to this term. But it is evidently not applicable to the product $0\cdot 5$ or $(-3)\cdot 5$: can you imagine a sum with zero or with minus three terms?

However, we may exchange the factors:

$$5 \cdot 0 = 0 + 0 + 0 + 0 + 0 = 0,$$

$$5 \cdot (-3) = (-3) + (-3) + (-3) + (-3) + (-3) = -15.$$

So if we want the product to be independent of the order of factors (as it was for positive numbers) we must agree that

$$0 \cdot 5 = 0, \quad (-3) \cdot 5 = -15.$$

Now let us consider the product $(-3) \cdot (-5)$. Is it equal to -15 or to $+15$? Both answers may have advocates. From one point of view, even one negative factor makes the product negative – so if both factors are negative the product has a very strong reason to be negative. From the other point of view, in the table

$3 \cdot 5 = +15$	$3 \cdot (-5) = -15$
$(-3) \cdot 5 = -15$	$(-3) \cdot (-5) = ?$

we already have two minuses and only one plus; so the "equal opportunities" policy requires one more plus. So what?

Of course, these "arguments" are not convincing to you. School education says very definitely that minus times minus is plus. But imagine that your small brother or sister asks you, "Why?" (Is it a caprice of the teacher, a law adopted by Congress, or a theorem that can be proved?) You may try to answer this question using the following example:

$3 \cdot 5 = 15$	Getting five dollars three times is getting fifteen dollars.
$3 \cdot (-5) = -15$	Paying a five-dollar penalty three times is a fifteen-dollar penalty.
$(-3) \cdot 5 = -15$	Not getting five dollars three times is not getting fifteen dollars.
$(-3) \cdot (-5) = 15$	Not paying a five-dollar penalty three times is getting fifteen dollars.

Another explanation. Let us write the numbers

$$1,\ 2,\ 3,\ 4,\ 5, \ldots$$

and the same numbers multiplied by three:

$$3,\ 6,\ 9,\ 12,\ 15, \ldots$$

Each number is bigger than the preceding one by three. Let us write the same numbers in the reverse order (starting, for example, with 5 and 15):

$$\begin{array}{ccccc} 5, & 4, & 3, & 2, & 1 \\ 15, & 12, & 9, & 6, & 3 \end{array}$$

Now let us continue both sequences:

$$\begin{array}{ccccccccccc} 5, & 4, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & -4, & -5, \ldots \\ 15, & 12, & 9, & 6, & 3, & 0, & -3, & -6, & -9, & -12, & -15, \ldots \end{array}$$

Here -15 is under -5, so $3 \cdot (-5) = -15$; plus times minus is minus.

Now repeat the same procedure multiplying $1, 2, 3, 4, 5, \ldots$ by -3 (we know already that plus times minus is minus):

$$\begin{array}{ccccc} 1, & 2, & 3, & 4, & 5 \\ -3, & -6, & -9, & -12, & -15 \end{array}$$

Each number is three units less than the preceding one. Now write the same numbers in the reverse order:

$$\begin{array}{ccccc} 5, & 4, & 3, & 2, & 1 \\ -15, & -12, & -9, & -6, & -3 \end{array}$$

and continue:

$$\begin{array}{ccccccccccc} 5, & 4, & 3, & 2, & 1, & 0, & -1, & -2, & -3, & -4, & -5, \ldots \\ -15, & -12, & -9, & -6, & -3, & 0, & 3, & 6, & 9, & 12, & 15, \ldots \end{array}$$

Now 15 is under -5; therefore $(-3) \cdot (-5) = 15$.

Probably this argument would be convincing for your younger brother or sister. But you have the right to ask: So what? Is it possible to *prove* that $(-3) \cdot (-5) = 15$?

Let us tell the whole truth now. Yes, it is possible to prove that $(-3) \cdot (-5)$ *must be* 15 *if* we want the usual properties of addition, subtraction, and multiplication that are true for positive numbers to remain true for any integers (including negative ones).

Here is the outline of this proof: Let us prove first that $3 \cdot (-5) = -15$. What is -15? It is a number opposite to 15, that is, a number that produces zero when added to 15. So we must prove that

$$3 \cdot (-5) + 15 = 0.$$

Indeed,

$$3 \cdot (-5) + 15 = 3 \cdot (-5) + 3 \cdot 5 = 3 \cdot (-5 + 5) = 3 \cdot 0 = 0.$$

(When taking 3 out of the parentheses we use the law $ab+ac = a(b+c)$ for $a = 3$, $b = -5$, $c = 5$; we assume that it is true for all numbers, including negative ones.) So $3 \cdot (-5) = -15$. (The careful reader will ask why $3 \cdot 0 = 0$. To tell you the truth, this step of the proof is omitted – as well as the whole discussion of what zero is.)

Now we are ready to prove that $(-3) \cdot (-5) = 15$. Let us start with

$$(-3) + 3 = 0$$

and multiply both sides of this equality by -5:

$$((-3) + 3) \cdot (-5) = 0 \cdot (-5) = 0.$$

Now removing the parentheses in the left-hand side we get

$$(-3) \cdot (-5) + 3 \cdot (-5) = 0,$$

that is, $(-3) \cdot (-5) + (-15) = 0$. Therefore, the number $(-3) \cdot (-5)$ is opposite to -15, that is, is equal to 15. (This argument also has gaps. We should prove first that $0 \cdot (-5) = 0$ and that there is only one number opposite to -15.)

15 Dealing with fractions

If somebody asks you to compare the fractions

$$\frac{3}{5} \quad \text{and} \quad \frac{9}{15},$$

you would answer immediately that they are equal:

$$\frac{9}{15} = \frac{3 \cdot 3}{3 \cdot 5} = \frac{3}{5}.$$

But what would you say now: Are the fractions

$$\frac{221}{391} \quad \text{and} \quad \frac{403}{713}$$

equal or not?

If you remember the multiplication table for two-digit numbers, you would say immediately that they are equal:

$$\frac{221}{391} = \frac{17 \cdot 13}{17 \cdot 23} = \frac{13}{23} = \frac{31 \cdot 13}{31 \cdot 23} = \frac{403}{713}$$

But what are we to do if we do not remember this multiplication table? Then we should find the common denominator for the two fractions,

$$\frac{221}{391} = \frac{221 \cdot 713}{391 \cdot 713} \quad \text{and} \quad \frac{403}{713} = \frac{403 \cdot 391}{713 \cdot 391}$$

and compare numerators,

$$\begin{array}{r} 713 \\ \underline{221} \\ 713 \\ 1426 \\ \underline{1426} \\ 157573 \end{array} \qquad \begin{array}{r} 391 \\ \underline{403} \\ 1173 \\ \underline{1564} \\ 157573 \end{array}$$

After that we would know that the fractions are equal but would never discover that in fact they are equal to $13/23$.

Problem 39. Which is bigger, $1/3$ or $2/7$?

Solution. $1/3 = 7/21$, $2/7 = 6/21$, so $1/3 > 2/7$.

The real-life version of this problem says, “Which is better, one bottle for three or two bottles for seven?” It suggests another solution: One bottle for three is equivalent to getting two bottles for six (and not for seven), so $1/3 > 2/7$. In scientific language, we found the “common numerator” instead of the common denominator:

$$\frac{1}{3} = \frac{2}{6} > \frac{2}{7}.$$

Problem 40. Which of the fractions

$$\frac{10001}{10002} \quad \text{and} \quad \frac{100001}{100002}$$

is bigger?

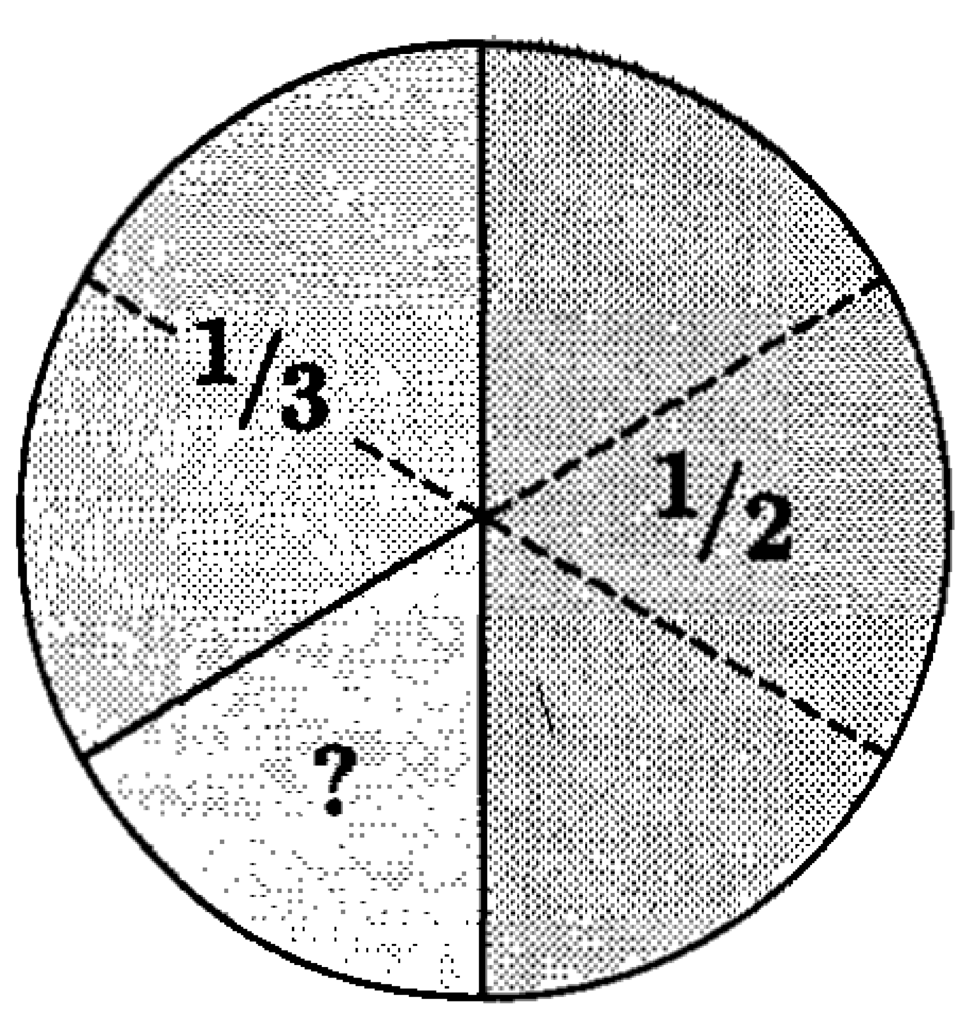
Hint. Both fractions are less than 1. What is the difference between them and 1?

Problem 41. Which of the fractions

$$\frac{12345}{54321} \quad \text{and} \quad \frac{12346}{54322}$$

is bigger?

Finding a common denominator is a traditional problem in teaching arithmetic. How much pie remains for you if your brother wants one-half and your sister wants one-third? The answer to this question is explained by the following picture:



Generally speaking, you need to find a common denominator when adding fractions. It is a horrible error (which, of course, you avoid) to add numerators and denominators separately:

$$\frac{2}{3}+\frac{5}{7} \longrightarrow \frac{2+5}{3+7}=\frac{7}{10}\,.$$

Instead of the sum this operation gives you something in between the two fractions you started with ($7/10 = 0.7$ is between $2/3 = 0.666\ldots$ and $5/7 = 0.714285\ldots$).

This is easy to understand in a real-life situation. Assume that one team has two bottles for three people ($2/3$ for each) and the other team has five bottles for seven people (5/7 for each). After they meet they have something in between $(2+5$ bottles for $3+7$ people).

Problem 42. Fractions $\frac{a}{b}$ and $\frac{c}{d}$ are called neighbor fractions if their difference $\frac{ad-bc}{bd}$ has numerator ± 1, that is, $ad - bc = \pm 1$. Prove that

(**a**) in this case neither fraction can be simplified (that is, neither has any common factors in numerator and denominator);

(**b**) if $\frac{a}{b}$ and $\frac{c}{d}$ are neighbor fractions, then $\frac{a+b}{c+d}$ is between them and is a neighbor fraction for both $\frac{a}{b}$ and $\frac{c}{d}$; moreover,

(**c**) no fraction $\frac{e}{f}$ with positive integer e and f such that $f < b+d$ is between $\frac{a}{b}$ and $\frac{c}{d}$.

Problem 43. A stick is divided by red marks into 7 equal segments and by green marks into 13 equal segments. Then it is cut into 20 equal pieces. Prove that any piece (except the two end pieces) contains exactly one mark (which may be red or green).

Solution. End pieces carry no marks because $\frac{1}{20}$ is smaller than $\frac{1}{7}$ and $\frac{1}{13}$. We have 18 other pieces – and it remains to prove that none of them can have more than one mark. (We have 18 marks – 6 red and 12 green – so no piece will be left without a mark.) Red marks correspond to numbers of the form $\frac{k}{7}$, green marks correspond to numbers of the form $\frac{l}{13}$. A fraction

$$\frac{k+l}{7+13} = \frac{k+l}{20}$$

is between them and is a cut point dividing these marks. Therefore, two marks of different colors cannot belong to the same piece. Two marks of the same color also cannot appear on one piece because the distance between them (either $1/7$ or $1/13$) is bigger than the piece length $1/20$.)

Problem 44. What is better, to get five percent of seven billion or seven percent of five billion?

Problem 45. How can you cut from a 2/3-meter-long string a piece of length 1/2 meter, without having a meter stick?

Solution. A piece of length 1/2 m constitutes three-fourths of the whole string:

$$\frac{3}{4} \cdot \frac{2}{3} = \frac{2}{4} = \frac{1}{2}$$

and you need to cut off one-fourth of the string.

16 Powers

In the sequence of numbers

$$2, 4, 8, 16, \ldots$$

each number is twice as large as the preceding one:

$$\begin{aligned} 4 &= 2 \cdot 2 \\ 8 &= 4 \cdot 2 = 2 \cdot 2 \cdot 2 \text{ (3 factors)} \\ 16 &= 8 \cdot 2 = 2 \cdot 2 \cdot 2 \cdot 2 \text{ (4 factors)} \\ &\ldots \end{aligned}$$

Mathematicians use the following useful notation:

$$\begin{aligned} 2 \cdot 2 &= 2^2 \\ 2 \cdot 2 \cdot 2 &= 2^3 \\ 2 \cdot 2 \cdot 2 \cdot 2 &= 2^4 \\ &\ldots \end{aligned}$$

so, for example, $2^6 = 64$.

Now the sequence $2, 4, 8, 16, \ldots$ can be written as $2, 2^2, 2^3, 2^4, \ldots$. We read a^n as "a to the n-th power" or "the n-th power of a"; a is called the *base*, and n is called an *exponent*.

There are special names for a^2 and a^3. They are "a squared" and "a cubed", respectively. (A square with side a has area a^2; a cube with edge a has volume a^3.)

Problem 46. Compute: (a) 2^{10}; (b) 10^3; (c) 10^7.

Problem 47. How many decimal digits do you need to write down 10^{1000}?

Astronomers use powers of 10 to write big numbers in a short form. For example, the speed of light is about 300,000 kilometers per second $= 3 \cdot 10^5$ km/s $= 3 \cdot 10^8$ m/s $= 3 \cdot 10^{10}$ cm/s.

Problem 48. In astronomy the distance covered by light in one year is called a *light-year.* What is the distance (approximately) between the Sun and the closest star measured in meters if it is about 4 light-years?

17 Big numbers around us

The number of molecules in one gram of water	$\simeq 3 \cdot 10^{22}$
The radius of Earth	$\simeq 6 \cdot 10^6$ m
The distance between Earth and the Moon	$\simeq 4 \cdot 10^8$ m
The distance between Earth and the Sun (the "astronomical unit")	$\simeq 1.5 \cdot 10^{11}$ m
The radius of the part of the universe observed up to now	$\simeq 10^{26}$ m
The mass of Earth	$\simeq 6 \cdot 10^{24}$ kg
The age of Earth	$\simeq 5 \cdot 10^9$ years
The age of the universe	$\simeq 1.5 \cdot 10^{10}$ years
The number of people on Earth	$\simeq 5 \cdot 10^9$
The average duration of a human life	$\simeq 2 \cdot 10^9$ seconds

Remark. When speaking of big numbers you must keep in mind that the same quantity may be big or small, depending on the unit you choose. For example, the distance between Earth and the Sun, measured in light-years, is about 0.000015 lt-yr, or, in meters (as seen from the table above), $1.5 \cdot 10^{11}$ m.

We shall see later that not only big numbers but also small numbers can be written conveniently using powers.

Programmers prefer to deal with powers of 2 (and not of 10). It turns out that $2^{10} = 1024$ is rather close to $1000 = 10^3$. So the prefix *kilo*, which usually means 1000 (1 kilogram = 1000 grams, 1 kilometer

= 1000 meters, etc.), means "1024" in programming: 1 kilobyte is 1024 bytes.

Problem 49. (a) How many decimal digits do you need to write down 2^{20}? (b) How many for the number 2^{100}? (c) Draw the graph showing how the number of decimal digits in 2^n depends on n.

(To answer the last question, the number of decimal digits in 2^n is approximately $0.3\,n$: $2^{10} \simeq 10^{0.3 \cdot 10}$, $2^n \simeq 10^{0.3\,n}$. Remember this when studying logarithms.)

Many types of pocket calculators use powers of 10 to show the product of two big numbers. For example,

$$370{,}000 \cdot 2{,}100{,}000 = 7.77 \cdot 10^{11},$$

but on the screen you do not see the dot and the base 10, just

`7.77     11`   or   `7.77     E11`

because of the screen limitations. In the usual form

`777000000000`

the answer would overflow the calculator screen.

18 Negative powers

We have seen the sequence of powers of 2:

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ ..$$

Now let us start with some number of the sequence (for example, 128) and write it in the reverse order:

$$128,\ 64,\ 32,\ 16,\ 8,\ 4,\ 2\,.$$

In the first sequence each number was two times bigger than the preceding one; in the second each number is two times smaller than the preceding one. Let us continue this sequence:

$$128,\ 64,\ 32,\ 16,\ 8,\ 4,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots$$

The sequence

$$2,\ 4,\ 8,\ 16,\ 32,\ 64,\ 128,\ \ldots$$

could be written as

$$2,\ 2^2,\ 2^3,\ 2^4,\ 2^5,\ 2^6,\ 2^7,\ \ldots.$$

In reverse order,

$$128,\ 64,\ 32,\ 16,\ 8,\ 4,\ 2$$

could be written as

$$2^7,\ 2^6,\ 2^5,\ 2^4,\ 2^3,\ 2^2,\ 2.$$

The analogy suggests the following continuation:

$$\begin{array}{cccccccccccc} 128, & 64, & 32, & 16, & 8, & 4, & 2, & 1, & \frac{1}{2}, & \frac{1}{4}, & \frac{1}{8}, & \frac{1}{16}, \ . \\ 2^7, & 2^6, & 2^5, & 2^4, & 2^3, & 2^2, & 2^1. & 2^0, & 2^{-1}, & 2^{-2}, & 2^{-3}, & 2^{-4} \ ,. \end{array}$$

This notation is widely used. So, for example,

$$2^3 = 8, \quad 2^1 = 2, \quad 2^0 = 1, \quad 2^{-1} = \frac{1}{2}, \quad 2^{-2} = \frac{1}{4}, \quad 2^{-3} = \frac{1}{8}, \text{ etc.}$$

When we spoke about powers before we said that 2^3 is "2 used 3 times as a factor" and 2^5 is "2 used 5 times as a factor". We can even say that 2^1 is "2 used once as a factor", but for 2^0 or 2^{-1} such an explanation cannot be taken seriously. It is just an agreement between mathematicians to understand 2^{-n} (for positive integer n) as $\frac{1}{2^n}$.

We hope that this agreement seems rather natural to you. Later we shall see that it is convenient and – in a sense – unavoidable.

Problem 50. Write down (**a**) 10^{-1}; (**b**) 10^{-2}; (**c**) 10^{-3} as decimal fractions.

19 Small numbers around us

1 cm	$= 10^{-2}$ m
1 mm	$= 10^{-3}$ m
1 μm	$= 10^{-6}$ m
1 nanometer	$= 10^{-9}$ m
1 angstrom	$= 10^{-10}$ m
The mass of a water molecule	$\simeq 3 \cdot 10^{-23}$ g
The size of a living cell	$\simeq$ 15 to $350 \cdot 10^{-9}$ m
The size at which modern physical laws become inapplicable (the "elementary length", as physicists say)	$\simeq 10^{-31}$ cm
The wavelength of red light	$\simeq 7 \cdot 10^{-7}$ m

As we have said already, there is no difference, in principle, between "big" and "small" numbers. For example, Earth's radius is about $6 \cdot 10^3$ km and at the same time about $4 \cdot 10^{-5}$ astronomical units.

Now let us return to the general definition of powers.

Definition. For positive integers n,

$$\begin{aligned} a^n &= a \cdot a \cdots a \quad (n \text{ times}) \\ a^{-n} &= \frac{1}{a^n} \\ a^0 &= 1 \end{aligned}$$

Problem 51. Is the equality $a^{-n} = \frac{1}{a^n}$ valid for negative n and for $n = 0$?

Is it possible to *prove* that $a^{-n} = \frac{1}{a^n}$? No, because the notation a^{-n} makes no sense without an agreement (called a *definition* by mathematicians). If suddenly all mathematicians change their mind and agree to understand a^{-n} in another way, then the equality $a^{-n} = \frac{1}{a^n}$ would be false. But you may be sure that this would never happen because nobody wants to violate such a convenient agreement. We would get into a mess if we did so.

Our notation allows us to write the long expression

$$2 \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot c \cdot c \cdot d$$

in the shorter form

$$2a^4b^3c^2d$$

and also rewrite

$$\frac{2 \cdot a \cdot a \cdot a \cdot a \cdot c \cdot c}{b \cdot b \cdot b \cdot d}$$

as

$$2a^4b^{-3}c^2d^{-1}.$$

Problem 52. Write the short form for the following expressions:

(a) $a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b$

(b) $\dfrac{2 \cdot a \cdot a \cdot a}{b \cdot b}$

Answer: (a) $a^{10}b^4$; (b) $2a^3b^{-2}$.

Problem 53. Rewrite using only positive powers:

$$\text{(a) } a^3b^{-5}; \qquad \text{(b) } a^{-2}b^{-7}.$$

Answer: (a) $\dfrac{a^3}{b^5}$; (b) $\dfrac{1}{a^2b^7}$.

20 How to multiply a^m by a^n, or why our definition is convenient

It is easy to multiply a^m by a^n if m and n are positive. For example,

$$a^5 \cdot a^3 = \underbrace{(a \cdot a \cdot a \cdot a \cdot a)}_{\text{5 times}} \cdot \underbrace{(a \cdot a \cdot a)}_{\text{3 times}} = a^8.$$

In general, $a^m \cdot a^n = a^{m+n}$ (indeed, a^m is a repeated m times and a^n is a repeated n times). Also

$$a^m \cdot a^1 = a^m \cdot a = a^{m+1}.$$

But the powers may also be negative. It turns out that our rule is valid in this case, too. For example, for $m = 5$, $n = -3$, it states that

$$a^5 \cdot a^{-3} = a^{5+(-3)} = a^2.$$

Let us check it: By definition, $a^5 \cdot a^{-3}$ is

$$a^5 \cdot \frac{1}{a^3} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a \cdot a} = a^2.$$

More pedantic readers would ask us to check also that

$$a^{-5} \cdot a^3 = a^{-5+3} = a^{-2}.$$

O.K. By definition,

$$a^{-5} \cdot a^3 = \frac{1}{a^5} \cdot a^3 = \frac{a^3}{a^5} = \frac{1}{a^2} = a^{-2}.$$

Even more pedantic readers would remember that both numbers m and n may be negative and ask to check, for example, that

$$a^{-5} \cdot a^{-3} = a^{(-5)+(-3)} = a^{-8}.$$

Indeed,

$$a^{-5} \cdot a^{-3} = \frac{1}{a^5} \cdot \frac{1}{a^3} = \frac{1}{a^8} = a^{-8}.$$

Don't relax – there are still other cases. One of the exponents (or even both) may be equal to zero, and a^0 was defined by a special agreement. So let us check that

$$a^m \cdot a^0 = a^{m+0} = a^m.$$

Indeed, $a^0 = 1$ by definition, so

$$a^m \cdot a^0 = a^m \cdot 1 = a^m.$$

Question. Is it necessary to consider the cases $m < 0$, $m = 0$ and $m > 0$ in the last argument separately?

Problem 54. Find a formula for $\dfrac{a^m}{a^n}$. Is your answer valid for all integers m and n?

21 The rule of multiplication for powers

When multiplying powers with the same base, you need to add exponents:

$$\boxed{a^m \cdot a^n = a^{m+n}}$$

This rule can be used to multiply small and big numbers in a convenient way. For example, to multiply $2 \cdot 10^7$ and $3 \cdot 10^{-11}$ we multiply 2 and 3 and add 7 and -11:

$$(2 \cdot 10^7) \cdot (3 \cdot 10^{-11}) = (2 \cdot 3) \cdot (10^7 \cdot 10^{-11}) = 6 \cdot 10^{7+(-11)} = 6 \cdot 10^{-4}.$$

This method is used in computers (but with base 2 instead of 10).

Problem 55. (a) You know that $2^{1001} \cdot 2^n = 2^{2000}$. What is n?

(b) You know that $2^{1001} \cdot 2^n = 1/4$. What is n?

(c) Which is bigger: 10^{-3} or 2^{-10}?

(d) You know that $\dfrac{2^{1000}}{2^n} = 2^{501}$. What is n?

(e) You know that $\dfrac{2^{1000}}{2^n} = 1/16$. What is n?

(f) You know that $4^{100} = 2^n$. What is n?

(g) You know that $2^{100} \cdot 3^{100} = a^{100}$. What is a?

(h) You know that $(2^{10})^{15} = 2^n$. What is n?

We said earlier that the definition of negative powers is in a sense unavoidable. Now we shall explain what we mean. Assume that we want to define negative power in some way, but want to keep the rule $a^{m+n} = a^m \cdot a^n$ true for all m and n. It turns out that the only way to do so is to follow our definition. Indeed, for $n = 0$ we must have $a^m \cdot a^0 = a^{m+0}$, that is, $a^m \cdot a^0 = a^m$. Therefore, $a^0 = 1$. But then $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$ implies that $a^{-n} = 1/a^n$.

What do we get if the power is used as a base for another power? For example,

$$(a^2)^3 = \underbrace{a^2 \cdot a^2 \cdot a^2}_{3 \text{ times}} = (a \cdot a) \cdot (a \cdot a) \cdot (a \cdot a) = a^6.$$

Similarly,

$$\boxed{(a^m)^n = a^{m \cdot n}}$$

for any positive m, n. And again our conventions "think for us": the same formula is also true for negative m and n. For example,

$$(a^{-2})^3 = \left(\frac{1}{a^2}\right)^3 = \frac{1}{a^2} \cdot \frac{1}{a^2} \cdot \frac{1}{a^2} = \frac{1}{a^6} = a^{-6} = a^{(-2)\cdot 3}.$$

Problem 56. Check this formula for other combinations of signs (if $m > 0$, $n < 0$; if both m and n are negative; if one of them is equal to zero).

The last formula about powers:

$$\boxed{(ab)^n = a^n \cdot b^n}$$

Problem 57. Check this formula for positive and negative integers n.

Problem 58. What is $(-a)^{775}$? Is it a^{775} or $-a^{775}$?

Problem 59. Invent a formula for $\left(\frac{a}{b}\right)^n$.

Now a^n is defined for any integer n (positive or not) and for any a. But that is not the end of the game, because n may be a number that is not an integer.

Problem 60. Give some suggestions: What might $4^{1/2}$ be? And $27^{1/3}$? Motivate your suggestions as well as you can.

The definition of $a^{m/n}$ will be given later. (But that also is not the last possible step.)

22 Formula for short multiplication: The square of a sum

As we have seen already,

$$(a + b)(m + n) = am + an + bm + bn$$

(to multiply two sums you must multiply each term of the first sum by each term of the second sum and then add all the products). Now consider the case when the letters inside the parentheses are the same:

$$(a + b)(a + b) = aa + ab + ba + bb.$$