1 Introduction

This book is about algebra. This is a very old science and its gems have lost their charm for us through everyday use. We have tried in this book to refresh them for you.

The main part of the book is made up of problems. The best way to deal with them is: Solve the problem by yourself – compare your solution with the solution in the book (if it exists) – go to the next problem. However, if you have difficulties solving a problem (and some of them are quite difficult), you may read the hint or start to read the solution. If there is no solution in the book for some problem, you may skip it (it is not heavily used in the sequel) and return to it later.

The book is divided into sections devoted to different topics. Some of them are very short, others are rather long.

Of course, you know arithmetic pretty well. However, we shall go through it once more, starting with easy things.

2 Exchange of terms in addition

Let's add 3 and 5:

$$3 + 5 = 8\,.$$

And now change the order:

$$5 + 3 = 8\,.$$

We get the same result. Adding three apples to five apples is the same as adding five apples to three – apples do not disappear and we get eight of them in both cases.

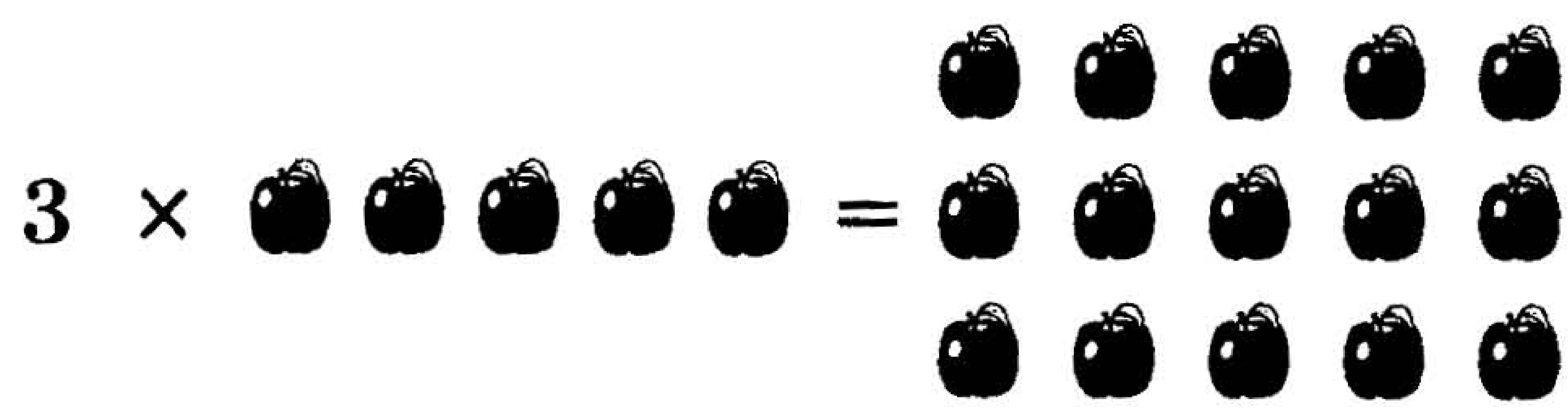
3 Exchange of terms in multiplication

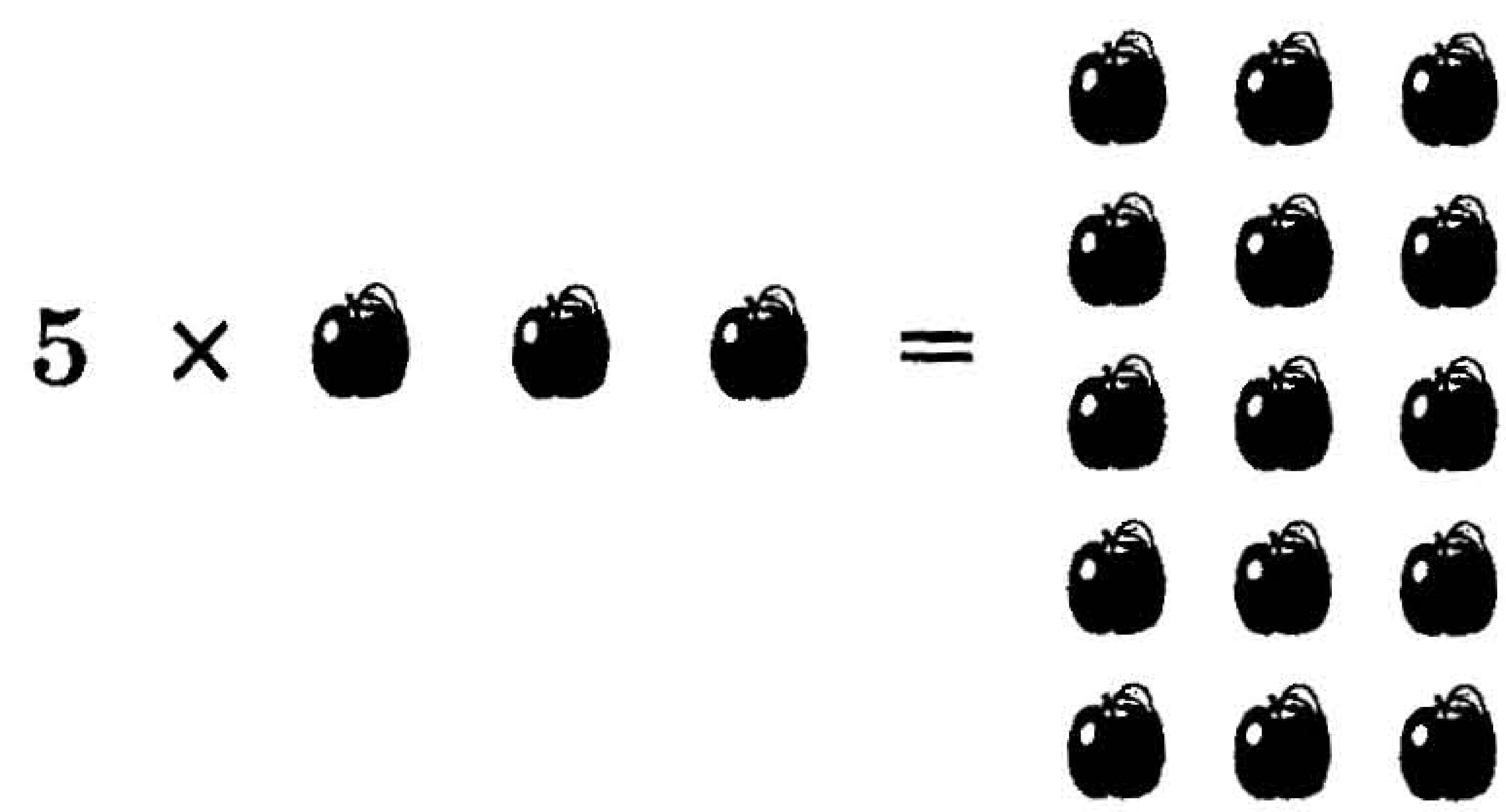
Multiplication has a similar property. But let us first agree on notation. Usually in arithmetic, multiplication is denoted as "$\times$". In algebra this sign is usually replaced by a dot "$\cdot$". We follow this convention.

Let us compare $3 \cdot 5$ and $5 \cdot 3$. Both products are 15. But it is not so easy to explain why they are equal. To give each of three boys five apples is not the same as to give each of five boys three apples – the situations differ radically.

One of the authors of this book asked a seven-year-old girl, "How much is two times four?" "Eight", she answered immediately. "And four times two?" She started thinking, trying to add $2 + 2 + 2 + 2$. A year later she would know very well that the product remains the same when we exchange factors and she would forget that it was not so evident before.

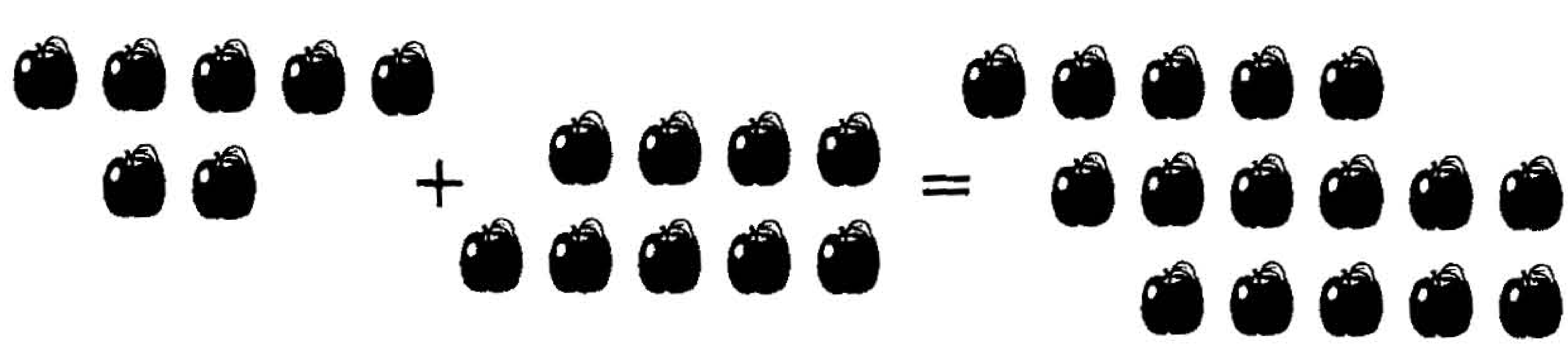
The simplest way to explain why $5 \cdot 3 = 3 \cdot 5$ is to show a picture:





4 Addition in the decimal number system

If we want to know how much $7+9$ is, we may draw 7 apples and then 9 apples near them:



and then count all the apples together: one, two, three, four, ..., fifteen, sixteen. We get $7 + 9 = 16$. This method can be applied for any numbers; however, you need a lot of patience to try it on, say, 137 and 268. So mathematicians invented other methods. One of them is the standard addition method used in the positional number system.

4 Addition in the decimal number system

In different countries and at different times, people used different notations for numbers, and entire books are written about them. We are so used to the familiar decimal number system using the digits $0, 1, 2, \ldots, 8, 9$ that we don't realize how unbelievably convenient this convention has proved to be. Even the possibility of writing down very big numbers quickly was not self-evident for ancient people. A great mathematician of ancient Greece, Archimedes, even wrote a book called *The Sand Reckoner.* The main point of the book was to show that it is possible to write down the number that is greater than the number of sand particles filling the sphere whose radius is the distance between Earth and the stars.

Now the decimal number system has no rivals – except the binary number system, which is popular among computers, not people. This binary system has only two digits, 0 and 1 – but numbers have more digits. The computer does not worry about the length of numbers, but still wishes to keep rules of operation as simple as possible.

We shall speak about the binary system in another section, but now we return to our ordinary decimal system and to the addition method. We shall not explain it to you once more – you know it without us. Let us solve some problems instead.

Problem 1. Several digits "8" are written and some "+" signs are inserted to get the sum 1000. Figure out how it is done. (For example, if we try $88 + 88 + 8 + 8 + 88$, we fail because we get only 280 instead of 1000.)

Solution. Assume that

$$\begin{array}{r} \ldots 8 \\ \ldots. \\ \underline{\ldots 8} \\ 1000 \end{array}$$

We do not know how many rows are here nor how many digits are used in each number. But we do know that each number ends with "8" and that the last digit of the sum is zero. How many numbers do we need to get this zero? If we use only one number, we get 8. If we use two numbers, we get 6 $(8 + 8 = 16)$, etc. To get zero we need at least five numbers:

$$\begin{array}{r} \ldots 8 \\ \ldots 8 \\ \ldots 8 \\ \ldots 8 \\ \underline{\ldots 8} \\ 1000 \end{array}$$

After we get this zero, we keep "4" in mind because $8+8+8+8+8=40$. To get the next zero in the "tens place" from this "4", we need to add at least two 8's since $4+8+8=20$.

$$\begin{array}{r} 8 \\ 8 \\ 8 \\ ..88 \\ \underline{..88} \\ 1000 \end{array}$$

We keep "2" in mind and we need only one more "8" to get 10:

$$\begin{array}{r} 8 \\ 8 \\ 8 \\ 88 \\ \underline{888} \\ 1000 \end{array}$$

The problem is solved: $8+8+8+88+888=1000$.

Problem 2. In the addition example

$$\begin{array}{r} \mathrm{A\,A\,A} \\ \underline{\mathrm{B\,B\,B}} \\ \mathrm{A\,A\,A\,C} \end{array}$$

all A's denote some digit, all B's denote another digit and C denotes a third digit. What are these digits?

Solution. First of all A denotes 1 because no other digit can appear as a carry in the thousands position of the result. To find what B is let us ask ourselves: Do we get a (nonzero) carry adding the rightmost A and B? If we had no carry, we would get the same digit in the other two places (tens and hundreds), but this is not so. Therefore, the carry digit is not zero, and this is possible only if $\mathrm{B} = 9$. Therefore we get the answer:

$$\begin{array}{r} 111 \\ \underline{999} \\ 1110 \end{array}$$

5 The multiplication table and the multiplication algorithm

To compute the product of, say, 17 and 38, we may draw a picture of 17 rows, each containing 38 points, and then count all the points. But of course, nobody does this – we know an easier method of multiplying using the positional system.

This method (called the *multiplication algorithm*) is based on the multiplication table for digits and requires that you memorize the table. There is – sorry! – no way around it, and if, on being asked, "What is seven times eight?" in the middle of the night, you cannot answer "Fifty-six!" immediately, and instead try to add up seven eights half-asleep, we are unable to help you.

There is some good news, however. You don't need to memorize the product $17 \cdot 38$. Instead, you can compute it in two different ways:

$$\begin{array}{r} 17 \\ \underline{38} \\ 136 \\ \underline{51} \\ 646 \end{array} \qquad\qquad \begin{array}{r} 38 \\ \underline{17} \\ 266 \\ \underline{38} \\ 646 \end{array}$$

Both results are equal, though the intermediate results are different. A lucky coincidence, isn't it?

Here are some problems concerning multiplication.

Problem 3. A boy claims that he can multiply any three-digit number by 1001 instantly. If his classmate says to him "715" he gives the answer immediately. Compute this answer and explain the boy's secret.

Problem 4. Multiply 101010101 by 57.

Problem 5. Multiply 10001 by 1020304050.

Problem 6. Multiply 11111 by 1111.

Problem 7. A six-digit number having 1 as its leftmost digit becomes three times bigger if we take this digit off and put it at the end of the number. What is this number?

Solution. Look at the multiplication procedure:

$$\begin{array}{r} 1\,A\,B\,C\,D\,E \\ 3 \\ \hline A\,B\,C\,D\,E\,1 \end{array}$$

Here A,B,C,D and E denote some digits (we do not know whether all these digits are different or not). Digit E must be equal to 7, because among the products $3\times 0=0$, $3\times 1=3$, $3\times 2=6$, $3\times 3=9$, $3\times 4=12$, $3\times 5=15$, $3\times 6=18$, $3\times 7=21$, $3\times 8=24$, $3\times 9=27$ only $3\times 7=21$ has the last digit 1. So we get:

$$\begin{array}{r} 1\,A\,B\,C\,D\,7 \\ 3 \\ \hline A\,B\,C\,D\,7\,1 \end{array}$$

When multiplying 7 by 3 we get a carry of 2, so $3\times D$ must have its last digit equal to 5. This is possible only if $D=5$:

$$\begin{array}{r} 1\,A\,B\,C\,5\,7 \\ 3 \\ \hline A\,B\,C\,5\,7\,1 \end{array}$$

In the same way, we find that $C=8$, $B=2$, $A=4$. So we get the solution:

$$\begin{array}{r} 1\,4\,2\,8\,5\,7 \\ 3 \\ \hline 4\,2\,8\,5\,7\,1 \end{array}$$

6 The division algorithm

Division is the most complicated thing among all the four arithmetic operations. To make yourself confident, you may try the following problems.

Problem 8. Divide 123123123 by 123. (Check your answer by multiplication!)

Problem 9. Can you predict the remainder when $111\ldots1$ (100 ones) is divided by 1111111?

Problem 10. Divide $1000\ldots0$ (20 zeros) by 7.

Problem 11. While solving the two preceding problems you may have discovered that quotient digits (and remainders) became periodic:

$$\begin{array}{l}
\overbrace{142857}14\ldots \\
7\,\overline{)\,100000000\ldots} \\
\underline{7} \\
30 \\
\underline{28} \\
20 \\
\underline{14} \\
60 \\
\underline{56} \\
40 \\
\underline{35} \\
50 \\
\underline{49} \\
10 \\
\underline{7} \\
30 \\
\underline{28} \\
2\ldots
\end{array}$$

Is it just a coincidence, or will this pattern repeat?

Problem 12. Divide $2000\ldots000$ (20 zeros), $3000\ldots000$ (20 zeros), $4000\ldots000$ (20 zeros), etc. by 7. Compare the answers you get and explain what you see.

A multiplication fan may enjoy the following problem:

Problem 13. Multiply 142857 by 1, 2, 3, 4, 5, 6, 7, and look at the results. (It is easy to memorize these results and become a famous number cruncher who is able to multiply a random number, for example, 142857, by almost any digit!)

Problem 14. Try to invent similar tricks based on the division of $1000\ldots0$ by other numbers instead of 7.

7 The binary system

Problem 15. Find a generating rule, and write five or ten more lines:

0
1
10
11
100
101
110
111
1000
1001
1010
1011
1100
...

Problem 16. You have weights of 1, 2, 4, 8, and 16 grams. Show that it is possible to get any weight from 0 to 31 grams using the following table ("+" means "the weight is used", "−" means "not used"):

	A					B	C
	16	8	4	2	1		
0	−	−	−	−	−	00000	0
1	−	−	−	−	+	00001	1
2	−	−	−	+	−	00010	10
3	−	−	−	+	+	00011	11
4	−	−	+	−	−	00100	100
5	−	−	+	−	+	00101	101
6	−	−	+	+	−	00110	110
7	−	−	+	+	+	00111	111
8	−	+	−	−	−	01000	1000
9	−	+	−	−	+	01001	1001
10	−	+	−	+	−	01010	1010
11	−	+	−	+	+	01011	1011

...

We can replace "−" by 0 and "+" by 1 (column B) and omit the leading zeros (column C). Then we get the same result as in the preceding problem.

This table is called a conversion table between decimal and binary number systems:

Decimal	Binary
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111
8	1000
9	1001
10	1010
11	1011
12	1100
...	...

Problem 17. What corresponds to 14 in the right column? What corresponds to 10000 in the left column?

The binary system has an advantage: you don't need to memorize as many as 10 digits; two is enough. But it has a disadvantage also: numbers are too long. (For example, 1024 is 10000000000 in binary.)

Problem 18. How is 45 (decimal) written in the binary system?

Problem 19. What (decimal) number is written as 10101101 in binary?

Problem 20. Try the usual addition method in binary version:

$$
\begin{aligned}
1010 &\quad + \quad 101 = ?\\
1111 &\quad + \quad 1 = ?\\
1011 &\quad + \quad 1 = ?\\
1111 &\quad + \quad 1111 = ?
\end{aligned}
$$

Check your answers, converting all the numbers (the numbers being added and the sums) into the decimal system.

Problem 21. Try the usual subtraction algorithm in its binary version:

$$\begin{aligned} 1101 &- 101 = ? \\ 110 &- 1 = ? \\ 1000 &- 1 = ? \end{aligned}$$

Check your answers, converting all the numbers into the decimal system.

Problem 22. Now try to multiply 1101 and 1010 (in binary):

$$\begin{array}{r} 1101 \\ \underline{1010} \\ ???? \end{array}$$

Check your result, converting the factors and the product into the decimal system.

Hint: Here are two patterns:

$$\begin{array}{r} 1011 \\ \underline{11} \\ 1011 \\ \underline{1011} \\ 100001 \end{array} \qquad \begin{array}{r} 1011 \\ \underline{101} \\ 1011 \\ \underline{1011} \\ 110111 \end{array}$$

Problem 23. Divide 11011 (binary) by 101 (binary) using the ordinary method. Check your result, converting all numbers into the decimal system.

Hint: Here is a pattern:

$$\begin{array}{rl} \,110 & \leftarrow \text{ the quotient} \\ 100\overline{)11001} & \\ \underline{100} & \\ 100 & \\ \underline{100} & \\ 1 & \leftarrow \text{ the remainder} \end{array}$$

Problem 24. In the decimal system the fraction $1/3$ is written as $0.333\ldots$. What happens with $1/3$ in the binary system?

8 The commutative law

Let us return to the rule "exchange of terms in addition does not change the sum". It can be written as

$$\text{First term} + \text{Second term} = \text{Second term} + \text{First term}$$

or in short

$$\text{F.t.} + \text{S.t.} = \text{S.t.} + \text{F.t.}$$

But even this short form seems too long for mathematicians, and they use single letters such as a or b instead of "F.t." and "S.t.". So we get

$$\boxed{a + b = b + a}$$

The law "exchange of factors does not change the product" can be written now as

$$\boxed{a \cdot b = b \cdot a}$$

Here "$\cdot$" is a multiplication symbol. Often it is omitted:

$$ab = ba$$

The property $a + b = b + a$ is called the *commutative law for addition*; the property $ab = ba$ is called the *commutative law for multiplication.*

Remark. Sometimes it is impossible to omit the multiplication sign ($\cdot$) in a formula; for example, $3 \cdot 7 = 21$ is not the same as $37 = 21$. By the way, multiplication had good luck in getting different symbols: the notations $a \times b$, $a \cdot b$, ab, and `a*b` (in computer programming) are all used.

9 The associative law

Now let us add three numbers instead of two:

$$3 + 5 + 11 = 8 + 11 = 19\,.$$

But there is another way:

$$3 + 5 + 11 = 3 + 16 = 19\,.$$

Usually parentheses are used to show the desired order of operations:

$$(3+5)+11$$

means that we have to add 3 and 5 first, and

$$3+(5+11)$$

means that we have to add 5 and 11 first.

The result does not depend on the order of the operations. This fact is called the *associative law* by mathematicians. In symbols:

$$\boxed{(a+b)+c=a+(b+c)}$$

If you would like to have a real-life example, here it is. You can get sweet coffee with milk if you add milk to the coffee with sugar or if you add sugar to the coffee with milk. You get the same result – and this is the associative law:

$$(\text{sugar}+\text{coffee})+\text{milk} = \text{sugar}+(\text{coffee}+\text{milk})$$

Problem 25. Try it.

Problem 26. Add $357+17999+1$ without paper and pencil.

Solution. It is not so easy to add 357 and 17999. But if you add $17999+1$, you get 18000 and now it is easy to add 357:

$$357+(17999+1)=357+18000=18357.$$

Problem 27. Add $357+17999$ without paper and pencil.

Solution. $357+17999=(356+1)+17999=356+(1+17999)=356+18000=18356$.

Problem 28. Add $899+1343+101$.

Hint. Remember the commutative law.

Multiplication is also associative:

$$(a\cdot b)\cdot c=a\cdot(b\cdot c)$$

or, in short,

$$(ab)c=a(bc).$$

Problem 29. Compute $37\cdot 25\cdot 4$.

Problem 30. Compute $125\cdot 37\cdot 8$.

10 The use of parentheses

A pedant is completely right saying that a notation like

$$2 \cdot 3 \cdot 4 \cdot 5$$

has no sense until we fix the order of operations. Even if we agree not to permute the factors, we have a lot of possibilities:

$$\begin{aligned}
((2 \cdot 3) \cdot 4) \cdot 5 &= (6 \cdot 4) \cdot 5 = 24 \cdot 5 = 120 \\
(2 \cdot (3 \cdot 4)) \cdot 5 &= (2 \cdot 12) \cdot 5 = 24 \cdot 5 = 120 \\
(2 \cdot 3) \cdot (4 \cdot 5) &= 6 \cdot 20 = 120 \\
2 \cdot ((3 \cdot 4) \cdot 5) &= 2 \cdot (12 \cdot 5) = 2 \cdot 60 = 120 \\
2 \cdot (3 \cdot (4 \cdot 5)) &= 2 \cdot (3 \cdot 20) = 2 \cdot 60 = 120
\end{aligned}$$

Problem 31. Find all possible ways to put parentheses in the product $2 \cdot 3 \cdot 4 \cdot 5 \cdot 6$ (not changing the order of factors; see the example just shown). Try to invent a systematic way of searching so as not to forget any possibilities.

Problem 32. How many "(" and ")" symbols do you need to specify completely the order of operations in the product

$$2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdots 99 \cdot 100\,?$$

The parentheses are often omitted because the result is independent of the order of the operations. The reader may reconstruct them as he or she wishes.

The following problem shows what can be achieved by clever permutation and grouping.

Problem 33. Compute $1 + 2 + 3 + 4 + \cdots + 98 + 99 + 100$.

Solution. Group the 100 terms in 50 pairs: $1+2+3+4+\cdots+98+99+100 = (1+100)+(2+99)+(3+98)+\cdots+(49+52)+(50+51)$. Each pair has the sum 101. We have 50 pairs, so the total sum is $50 \cdot 101 = 5050$.

A legend says that as a schoolboy Karl Gauss (later a great German mathematician) shocked his school teacher by solving this problem instantly (as the teacher was planning to relax while the children were busy adding the hundred numbers).

11 The distributive law

There is one more law for addition and multiplication, called the *distributive law.* If two boys and three girls get 7 apples each, then the boys get $2 \cdot 7 = 14$ apples, the girls get $3 \cdot 7 = 21$ apples – and together they get

$$2 \cdot 7 + 3 \cdot 7 = 14 + 21 = 35$$

apples. The same answer can be computed in another way: each of $2 + 3 = 5$ children gets 7 apples, so the total number of apples is

$$(2 + 3) \cdot 7 = 5 \cdot 7 = 35\,.$$

Therefore,

$$(2 + 3) \cdot 7 = 2 \cdot 7 + 3 \cdot 7$$

and, in general,

$$(a + b) \cdot c = a \cdot c + b \cdot c$$

This property is called the *distributive law.* Changing the order of factors we may also write

$$c \cdot (a + b) = c \cdot a + c \cdot b$$

Problem 34. Compute $1001 \cdot 20$ without pencil and paper.

Solution. $1001 \cdot 20 = (1000 + 1) \cdot 20 = 1000 \cdot 20 + 1 \cdot 20 = 20{,}000 + 20 = 20{,}020$.

Problem 35. Compute $1001 \cdot 102$ without pencil and paper.

Solution. $1001 \cdot 102 = 1001 \cdot (100 + 2) = 1001 \cdot 100 + 1001 \cdot 2 = (1000 + 1) \cdot 100 + (1000 + 1) \cdot 2 = 100{,}000 + 100 + 2000 + 2 = 102{,}102$.

The distributive law is a rule for removing brackets or parentheses. Let us see how it is used to transform the product of two sums

$$(a + b)(m + n)\,.$$

The number $(m + n)$ is the sum of the two numbers m and n and can replace c in the distributive law above:

$$(a + b) \cdot \boxed{c} = a \cdot \boxed{c} + b \cdot \boxed{c}$$

$$(a + b) \cdot \boxed{m + n} = a \cdot \boxed{m + n} + b \cdot \boxed{m + n}$$

Now we remember that $\boxed{m+n}$ is the sum of m and n and continue:

$$\ldots = a(m+n) + b(m+n) = am + an + bm + bn\,.$$

The general rule: To multiply two sums you need to multiply each term of the first sum by each term of the second one and then add all the products.

Problem 36. How many additive terms would be in

$$(a+b+c+d+e)(x+y+z)$$

after we use this rule?

12 Letters in algebra

In algebra we gradually make more and more use of letters (such as $a, b, c, \ldots, x, y, z$, etc.). Traditionally the use of letters (x's) is considered one of the most difficult topics in the school mathematics curriculum. Many years ago primary school pupils studied "arithmetic" (with no x's) and secondary school pupils started with "algebra" (with x's). Later "arithmetic" was renamed "mathematics" and x's were introduced (and created a mess, some people would say).

We hope that you, dear reader, never had difficulties understanding "what all these letters mean", but we still wish to give you some advice. If you ever want to explain the meaning of letters to your classmates, brothers and sisters, your parents, or your children (some day), just say that the letters are abbreviations for words. Let us explain what we mean.

In the equality

$$a + b = b + a$$

the letters a and b mean "the first term" and "the second term". When we write $a+b=b+a$ we mean that any numbers substituted instead of a and b give a true assertion. Therefore, $a+b=b+a$ can be considered as a unified short version of the equalities $1+7=7+1$ or $1028+17=17+1028$ as well as infinitely many other equalities of the same type.

Another example of the use of letters:

Problem 37. A small vessel and a big vessel contain (together) 5 liters. Two small and three big vessels contain together 13 liters. What are the volumes of the vessels?

Solution. (The "arithmetic" one.) The small and big vessels together contain 5 liters. Therefore, two small vessels and two big vessels together contain 10 liters ($10 = 2 \cdot 5$). As we know, two small vessels and three big vessels contain 13 liters. So we get 13 liters instead of 10 by adding one big vessel. Therefore the volume of a big vessel is 3 liters. Now it is easy to find the volume of a small vessel: together they contain 5 liters, so a small vessel contains $5 - 3 = 2$ liters. Answer: The volume of a small vessel is 2 liters, the volume of a big vessel is 3 liters.

This solution can be shortened if we use "Vol.SV" instead of "Volume of a Small Vessel" and "Vol.BV" instead of "Volume of a Big Vessel". Thus, according to the statement of the problem,

$$\text{Vol.SV} + \text{Vol.BV} = 5,$$

therefore

$$2 \cdot \text{Vol.SV} + 2 \cdot \text{Vol.BV} = 10\,.$$

We know also that

$$2 \cdot \text{Vol.SV} + 3 \cdot \text{Vol.BV} = 13\,.$$

If we subtract the preceding equality from the last one we find that $\text{Vol.BV} = 3$. Now the first equality implies that $\text{Vol.SV} = 5 - 3 = 2$.

Now the only thing to do is to replace our "Vol.SV" and "Vol.BV" by standard unknowns x and y – and we get the standard "algebraic" solution of our problem. Here it is: Denote the volume of a small vessel by x and the volume of a big vessel by y. We get the following system of equations:

$$\begin{aligned} x + y &= 5 \\ 2x + 3y &= 13\,. \end{aligned}$$

Multiplying the first equation by 2 we get

$$2x + 2y = 10$$

and subtracting the last equation from the second equation of our system we get

$$y = 13 - 10 = 3\,.$$

Now the first equation gives

$$x = 5 - y = 5 - 3 = 2\,.$$

Answer: $x = 2$, $y = 3$.

Finally, one more example of the use of letters in algebra.

"Magic trick". Choose any number you wish. Add 3 to it. Multiply the result by 2. Subtract the chosen number. Subtract 4. Subtract the chosen number once more. You get 2, don't you?

Problem 38. Explain why this trick is successful.

Solution. Let us follow what happens with the chosen number (we denote it by x):

Choose the number you wish	x
add 3 to it	$x + 3$
multiply the result by 2	$2 \cdot (x + 3) = 2x + 6$
subtract the chosen number	$(2x + 6) - x = x + 6$
subtract 4	$(x + 6) - 4 = x + 2$
subtract the chosen number once more. You get 2.	$(x + 2) - x = 2$

13 The addition of negative numbers

It is easy to check that $3 + 5 = 8$: just take three apples, add five apples, and count all the apples together: "one, two, three, four, ..., seven, eight". But how can we check that $(-3) + (-5) = (-8)$ or that $3 + (-5) = (-2)$? Usually this is explained by examples like the following two: