If a cell complex X is simply connected, then for $n \geq 2$, $\pi_n(X, x_0)$ is a finitely generated abelian group, and so it is isomorphic to a direct product of finitely many copies of the infinite cyclic group $\mathbb{Z}$ with finitely many copies of some finite cyclic groups $\mathbb{Z}/(p_i)$.

Here are several examples.

EXAMPLE 3.13.

$$\pi_n(D^k) = 0, \quad k \geq 0; \qquad \pi_n(S^1) = 0, \quad n \geq 2;$$
$$\pi_n(S^n) \cong \mathbb{Z}, \quad \pi_n(S^k) = 0, \quad n < k;$$
$$\pi_n(\text{letters } \mathsf{A}, \mathsf{B}, \dots, \mathsf{Z}) = 0, \quad n \geq 2.$$

In order to show that $\pi_n(S^1) = 0$ $(n \geq 2)$, for instance, we regard S^1 as the quotient space of $\mathbb{R}$ under the identification of points that differ by integers. Then we can show that $\pi_n(S^1) \cong \pi_n(\mathbb{R})$, and hence our claim.

EXAMPLE 3.14.

$$\pi_n(S^1 \times S^1) = 0, \quad n \geq 2.$$

We identify $S^1 \times S^1$ with the quotient space of $\mathbb{R}^2$ under the identification of points which differ componentwise by integers. Then we can show that $\pi_n(S^1 \times S^1) \cong \pi_n(\mathbb{R})$, $n \geq 2$, and the conclusion follows.

EXAMPLE 3.15.

$$\pi_3(S^2) \cong \mathbb{Z}.$$

The Hopf map $f : S^3 \to S^2$ representing the generator $1 \in \mathbb{Z}$ will come aboard as the projection of some fiber bundle in Chapter Eight.

3.4. Homotopy invariance

Suppose we have a map

$$f : (X, x_0) \to (Y, y_0)$$

between two pairs (X, x_0) and (Y, y_0). Then f determines naturally an induced map

$$f_* : [(I^n, \partial I^n), (X, x_0)] \to [(I^n, \partial I^n), (Y, y_0)],$$

that sends the homotopy class generated by $g : (I^n, \partial I^n) \to (X, x_0)$ to the homotopy class of $f \circ g : (I^n, \partial I^n) \to (Y, y_0)$. It is easy to see from the definition that

$$f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0),$$

is a group homomorphism.

The following theorem will become evident as we recall the definition of homotopy equivalences (cf. §1.3).

THEOREM 3.16 (homotopy invariance). *If two pairs (X, x_0) and (Y, y_0) have the same homotopy type, then for each n we have a group isomorphism*

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0).$$

When a connected topological space X enjoys a very nice geometrical property called homogeneity (for instance, X is a manifold), there is a homeomorphism $h : (X, x_0) \to (X, x_1)$ for any pair of points x_0 and x_1 of X. This implies that there exists a homotopy equivalence between (X, x_0) and (X, x_1), and so we have group isomorphisms

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for all n.

Actually, for a connected cell complex X, not necessarily homogeneous, the pointed spaces (X, x_0) and (X, x_1) have the same homotopy type for any pair of points x_0, x_1, and hence we have an isomorphism

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for every n. The proof of homotopy type is not straightforward, however, and so we will directly establish isomorphisms of homotopy groups in the following

THEOREM 3.17. *If X is a connected topological space, then for any two points x_1 and x_1 we have a group isomorphism*

$$\pi_n(X, x_0) \cong \pi_n(X, x_1)$$

for each natural number n.

PROOF. Consider a continuous curve $x_t \in X$, $t \in [0, 1]$, which connects x_0 and x_1. We define a family of maps from I^n to X taking the boundary of I^n along this curve, which naturally defines a family of homomorphisms

$$h_t : \pi_n(X, x_0) \to \pi_n(X, x_t),$$

where $t \in [0, 1]$. Similarly we get homomorphisms

$$\hat{h}_t : \pi_n(X, x_t) \to \pi_n(X, x_0),$$

so that we have

$$h_1 \circ \hat{h}_1 = id, \quad \hat{h}_1 \circ h_1 = id.$$

□